18/11/2015 الأربعاء د. عرفة محاضرة $\boxed{7}$

* Circular Convolution :-

If the sequences (Discrete time Sequences) are periodic, then the Convolution is circular.

$$y(n) = x_1(n) \; Ⓝ \; x_2(n)$$

where $x_1(n)$ & $x_2(n)$ are two periodic sequences every N samples.

Ⓝ $\Rightarrow$ The symbol of circular convolution.

$$y(n) = x_1(n) \; Ⓝ \; x_2(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N \checkmark$$

$$= \sum_{n=0}^{N-1} x_1((n-m))_N \cdot x_2(m) \checkmark$$

$$= IDFT\left\{ \underbrace{X_1(K)}_{\text{DFT for } x_1(n)} \cdot \underbrace{X_2(K)}_{\text{DFT for } x_2(n)} \right\}$$

Example:- Compute the circular Convolution for:

$x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$

for $N=4$ (Find $x_1(n)$ ④ $x_2(n)$)

$$y(n) = x_1(n) \; ④ \; x_2(n) = \sum_{m=0}^{3} x_1(m) \, x_2((n-m))_4$$

	M	0	1	2	3	
	$x_1(m)$	2	1	2	1	
	$x_2(m)$	1	2	3	4	
$n=0$	$x_2((-m))_4$	1	4	3	2	$\Rightarrow y(0)$
$n=1$	$x_2((1-m))_4$	2	1	4	3	$\Rightarrow y(1)$
$n=2$	$x_2((2-m))_4$	3	2	1	4	$\Rightarrow y(2)$
$n=3$	$x_2((3-m))_4$	4	3	2	1	$\Rightarrow y(3)$

$y(0) = 2*1 + 1*4 + 2*3 + 1*2 = 14$

$y(1) = 2*2 + 1*1 + 2*4 + 1*3 = 16$

$y(2) = 14$

$y(3) = 16$

$y(n) = \{14, 16, 14, 16\}$

- Another Method

$y(n) = x_1(n) \; N \; x_2(n)$

$x_1(n)$

$$= \begin{pmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}_{\substack{N \times N \\ 4 \times 4}} \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}_{\substack{N \times 1 \\ 4 \times 1}}$$

$$= \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

For shifting $x_1(n)$

$$y(n) = \sum_{m=0}^{3} x_2(m) \, x_1((n-m))_4$$

$$x_1((-n))_4 \Longrightarrow$$

$$= \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

$$x_1((3-n))_4 \Longrightarrow$$

Another Solution:-

$y(n) = x_1(n) \; (4) \; x_2(n) = IDFT\,(X_1(k) \cdot X_2(k))$

$\boxed{1}$ Compute DFT for $x_1(n)$ & $x_2(n)$

$$X_1(k) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad X_2(k) = \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix}$$

$$X_1(k) \cdot X_2(k) = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$\boxed{2}$ Comput IDFT

$$y(n) = \begin{pmatrix} 14 \\ 16 \\ 14 \\ 16 \end{pmatrix}$$

Prove the following properties:-

$\boxed{1}$ $W_N^{k+N} = W_N^{k}$ $\qquad$ $\boxed{2}$ $W_N^{k+N/2} = -W_N^{k}$

$\boxed{3}$ $W_N^2 = W_{N/2}$ $\qquad$ $W_N = e^{-j\frac{2\pi}{N}}$ (twiddle factor)

$\boxed{1}$ $W_N^{k+N} = W_N^{k} * (W_N^{N}) \longrightarrow 1 \; ; \; W_N^N = e^{-j\frac{2\pi N}{N}} = e^{-j2\pi} = 1$

$\boxed{2}$ $W_N^{k+N/2} = W_N^{k} \, (W_N^{N/2}) \longrightarrow -1 \; ; \; W_N^{N/2} = e^{-j\frac{2\pi N}{2N}} = e^{-j\pi} = -1$

$\boxed{3}$ $W_{N/2} = e^{-j\frac{2\pi}{N/2}} = (e^{-j\frac{2\pi}{N}})^2 = W_N^2$

M

3

* Fast Fourier Transform (FFT)

FFT Algorithms are many algorithms used to reduce the complex computation for DFT

[1] Radix - 2 DIT FFT

splitting ← Decimation in time

For periodic sequence $x(n)$, the discrete fourier transform (DFT) is $X(K)$, which has periodic N samples

$$\underset{DTS}{x(n)} \xrightarrow[N]{DFT} \underset{DFT}{X(K)}$$

$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n)\, w_N^{Kn} \quad \text{where } w_N = e^{-j\frac{2\pi}{N}}}$$

⟹ The previous method

For Radix-2 DIT FFT algorithm:

$$X(K) = \underbrace{\sum_{n=\text{even}}}_{\frac{N}{2}\text{ point DFT}} + \underbrace{\sum_{n=\text{odd}}}_{\frac{N}{2}\text{ point DFT}}$$

$$X(K) = \sum_{n=0}^{N/2-1} \underbrace{x(2n)}_{\text{even Number seq.}} w_N^{2Kn} + \sum_{n=0}^{N/2-1} \underbrace{x(2n+1)}_{\text{odd Number seq.}} w_N^{K(2n+1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, w_{N/2}^{Kn} + \sum_{n=0}^{N/2-1} x(2n+1)\, w_{N/2}^{Kn}\, w_N^{K}$$

$$= \sum_{n=0}^{N/2-1} x(2n)\, w_{N/2}^{Kn} + w_N^{K} \sum_{n=0}^{N/2-1} x(2n+1)\, w_{N/2}^{Kn}$$

assume $f_1(n) = x(2n)$
$f_2(n) = x(2n+1)$

$$X(K) = \sum_{n=0}^{N/2-1} f_1(n)\, W_{N/2}^{kn} + W_N^{K} \sum_{n=0}^{N/2-1} W_{N/2}^{kn}$$

$$X(K) = F_1(K) + W_N^{K} F_2(K) \Rightarrow ① \quad \tfrac{N}{2} \text{ DFT points}$$

where $F_1(K) \Rightarrow$ DFT for $f_1(n) = x(2n)$
& $F_2(K) \Rightarrow$ // // $f_2(n) = x(2n+1)$

$F_1(K) \longrightarrow \frac{N}{2}$ point DFT
$F_2(K) \longrightarrow \frac{N}{2}$ point DFT

for eq (1) $\Rightarrow K = 0, 1, \ldots, \frac{N}{2} - 1 \Rightarrow X(K)$ has $\frac{N}{2}$ point DFT

$$X\left(K+\tfrac{N}{2}\right) = F\left(K+\tfrac{N}{2}\right) + W_N^{K+\frac{N}{2}} F_2\left(K+\tfrac{N}{2}\right)$$

$$X\left(K+\tfrac{N}{2}\right) = F_1(K) - W_N^{K} F_2(K)$$

$$K = 0, 1, \ldots \tfrac{N}{2} - 1$$

For large values of N, this operation is repeated until we reach to 2-point DFT computation

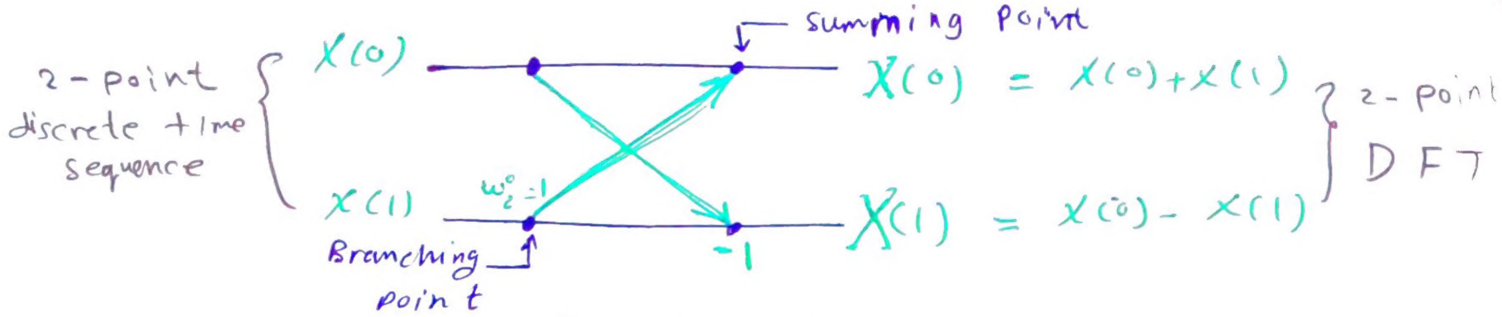
Example: $X(n) = \{x(0), x(1)\} \Rightarrow N = 2$

$$X(K) = \sum_{n=0}^{1} x(n)\, W_2^{kn}$$

$$= x(0)\, \underset{1}{W_2^{0}} + x(1)\, W_2^{K}$$

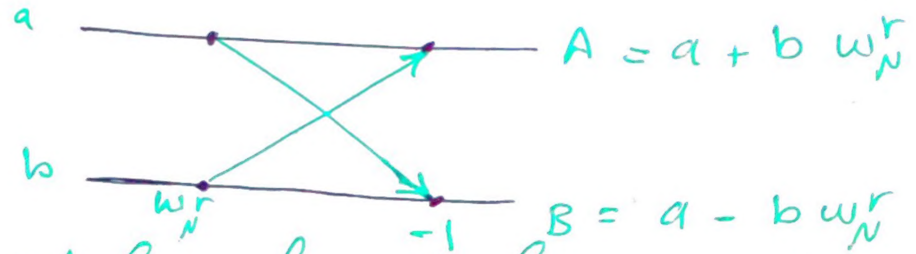
$K = 0 \Rightarrow X(K) = x(0) + x(1)$
$K = 1 \Rightarrow X(K) = x(0) - x(1)$

$\Longrightarrow$ Turn over



Butterfly Computation

In general :



General form for butterfly computation

Ex: FFT

for $N=4$

$$X(K) = F_1(K) + w_N^K F_2(K) \Rightarrow ①$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - w_N^K F_2(K) \Rightarrow ②$$

$$K = 0, 1, \cdots, \frac{N}{2}-1$$

where $f_1(n) = x(2n) \rightarrow$ even numbered seq.

$f_2(n) = x(2n+1) \rightarrow$ odd " "

$$X(K) = F_1(K) + w_4^k F_2(K) \longrightarrow ①$$

$$X(K+2) = F_1(K) - w_4^K F_2(K) \longrightarrow ②$$

$K = 0, 1$

$\Longrightarrow$ Turn over

eq (1)

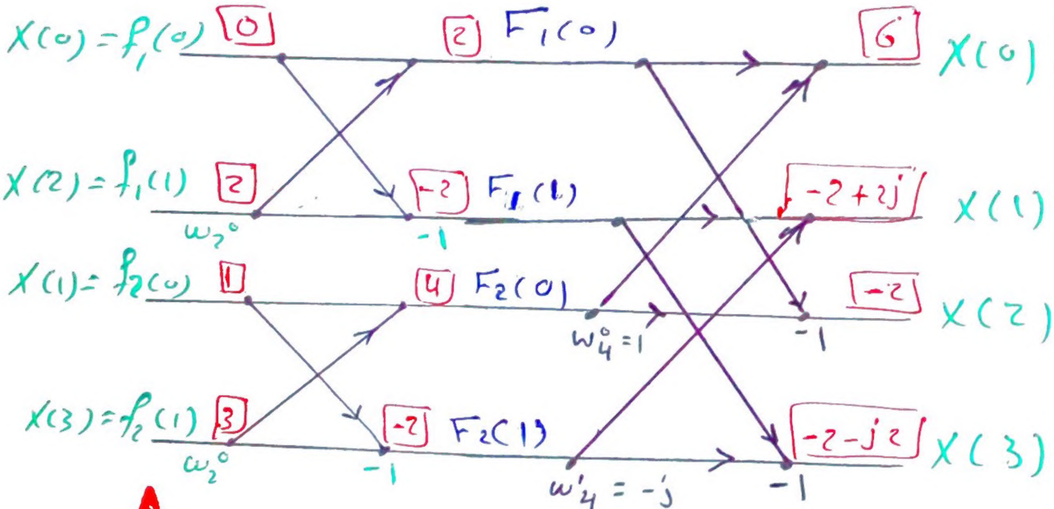
$K=0 \Rightarrow X(0) = F_1(0) + W_4^0 \, F_2(0)$
$K=1 \Rightarrow X(1) = F_1(1) + W_4^1 \, F_2(1)$ } 2-point DFT

eq (2)

$K=0 \Rightarrow X(2) = F_1(0) - W_4^0 \, F_2(0)$
$K=1 \Rightarrow X(3) = F_1(1) - W_4^1 \, F_2(1)$ } 2-point DFT

4-point DFT

اللون الأحمر حل للمثال في آخر الصفحة



General form for $N=4$

$N=4$;

$f_1(n) = x(2n) \Rightarrow n=0,1$
$f_1(n) = \{x(0), x(2)\}$ } even

$f_2(n) = x(2n+1) \Rightarrow n=0,1$
$f_2(n) = \{x(1), x(3)\}$ } odd

ex: find Radix-2 DIT FFT for

$x(n) = \{0, 1, 2, 3\}$

$X(K) \{6, -2+j2, -2, -2-2j\}$

الحل مسقط على الرسمة السابقة باللون الأحمر :